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Canonical scattering transformation in quantum mechanics

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Abstract. We quantise the classical canonical scattering transformation of Hunziker with the representation method of Moshinsky and Seligmann. This leads to a sheeted phase space characterised by the number of turns around the scatterer. The usual detection device projects on the trivial representation of the corresponding ambiguity group. This operation extracts the integer values of the angular momentum.

1. Introduction

Recently, following an idea of Hunziker (1968), Narnhofer and Thirring (1981) have proposed a nice classical picture of the S-matrix theory. In this approach the pertinent object is a generating function of the canonical map between the straight asymptotic trajectories before and after the collision. This generating function is simply twice the phase shift multiplied by $\hbar$.

The concept of generating functions of canonical transformations is old and forgotten among the physicist community. A new look at the geometrical significance of this formalism may be found in Amiet and Huguenin (1980) for example. The use of these generating functions for the calculation of matrix elements of unitary transformations (Amiet and Huguenin 1981) enhances its significance for quantum mechanics as already seen by Van Vleck (1928).

The quantisation of the classical result of Narnhofer and Thirring is, nevertheless, non-trivial. For example the phaseshifts are usually defined for integer values of the angular momentum, and the definition of the derivative is ambiguous, i.e. the deflection angle is not properly defined in quantum mechanics.

The reason for the difficulty is that the appropriate variables for the description of the scattering are the action-angle variables related to the conserved quantities instead of the original $(\boldsymbol{q}, \boldsymbol{p})$ phase space variables, a transformation which is not one-to-one.

A way to solve this kind of problem may be found in the work of Moshinsky and Seligmann (1980). The concept of ambiguity of a canonical transformation finds a natural application in the domain of scattering. Roughly speaking, the detector does not separate the contributions of the number of turns of the projectile around the scatterer. This is the physical ambiguity. The detector is sensitive to the coherent sum of contributions of an arbitrary number of turns. The result is a projection onto the trivial representation of the ambiguity group. This projection retains only the integer values of the angular momentum. In the discussion of Aharonov–Bohm scattering, Berry (1980) has already proposed a similar approach.

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In this paper, we deal with scattering in the plane by a central potential with all conditions guaranteeing the existence of the asymptotic states. We use the following conventions:

$$\sum_{l=-\infty}^{\infty} = \sum_{l} \qquad \int_{-\infty}^{\infty} \mathrm{d}\lambda = \int \mathrm{d}\lambda.$$

2. Classical scattering

Here we recall the results of Narnhofer and Thirring (1981) with special emphasis on the ambiguity related to the use of polar coordinates in the momentum plane. Consider the map

$$f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R} \times [0, 2\pi[\times \mathbb{R}^+$$
$$(\boldsymbol{q}, \boldsymbol{p}) \mapsto (t, \mathscr{L}; \chi, h)$$

defined by (figure 1)

$$t = m\,\boldsymbol{q} \cdot \boldsymbol{p}/\|\boldsymbol{p}\|^2 \qquad \mathscr{L} = q_x p_y - q_y p_x \qquad \chi = \tan^{-1} p_y/p_x \qquad h = \|\boldsymbol{p}\|^2/2m. \quad (2.1)$$

Except for $\|\boldsymbol{p}\| = 0$, the mapping exists and is one-to-one. But for the sake of quantisation, the edge of the domain of $(t, \mathscr{L}; \chi, h)$ presents some difficulties. Following Moshinsky and Seligmann (1977, 1979, 1980), we propose to extend the mapping f continuously in such a way that $(t, \mathscr{L}; \chi, h) \in \mathbb{R}^4$. To this end, we define the domains D_N which cover $\mathbb{R}^4$:

$$D_N = \mathbb{R} \times \mathbb{R} \times [N\pi, (N+2)\pi[\times (-1)^N \mathbb{R}^+ \qquad N \in \mathbb{Z}.$$

The mapping

$$a: D_N \to D_{N+1}$$
$$(t, \mathscr{L}; \chi, h) \mapsto (-t, \mathscr{L}; \chi + \pi, -h) \quad (2.2)$$

generates the group $\mathbb{Z}$ and the powers of a relate the D_N to one another. It follows

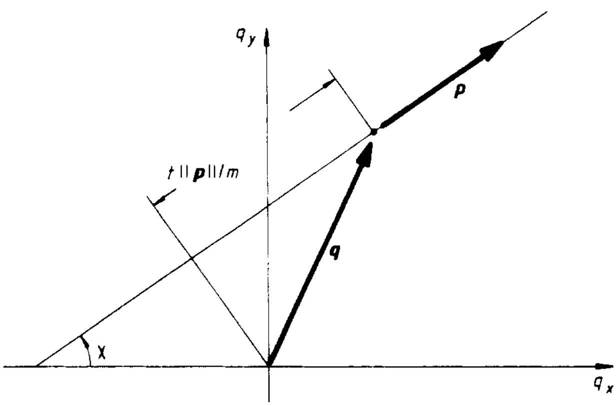

Figure 1. $(t, \mathscr{L}; \chi, h)$ are the appropriate variables for the description of the scattering, in particular for the time delay and the deflection angle.

that for all $(t, \mathscr{L}; \chi, h) \in \mathbb{R}^4$ there exists one (unique) $N \in \mathbb{Z}$ such that

$$a^{-N}(t, \mathscr{L}; \chi, h) \in D_0$$

(provided simply that $h \neq 0$). We propose now an extension F of f by introducing a sheeted phase space

$$F: \mathbb{Z} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^4 \qquad (N, \boldsymbol{q}, \boldsymbol{p}) \mapsto (t, \mathscr{L}; \chi, h) \in D_N$$

with

$$t = (-1)^N m\boldsymbol{q} \cdot \boldsymbol{p}/\|\boldsymbol{p}\|^2 \qquad \mathscr{L} = q_x p_y - q_y p_x \qquad \chi = \tan^{-1} p_y/p_x + N\pi$$
$$h = (-1)^N \|\boldsymbol{p}\|^2/2m. \tag{2.3}$$

The inverse transformation F^{-1} is given by

$$q_x = \frac{(-1)^N}{\sqrt{(-1)^N 2mh}}(2th \cos \chi + \mathscr{L} \sin \chi) \qquad p_x = (-1)^N \sqrt{(-1)^N 2mh} \cos \chi$$

$$q_y = \frac{(-1)^N}{\sqrt{(-1)^N 2mh}}(2th \sin \chi - \mathscr{L} \cos \chi) \qquad p_y = (-1)^N \sqrt{(-1)^N 2mh} \sin \chi$$

with N such that

$$(-1)^N = h/|h| \qquad \chi - N\pi \in [0, 2\pi[.$$

In this way we achieve a covering of $\mathbb{R}^2 \times \mathbb{R}^2$ where the subspace $\mathbb{R}^2 \times \{\boldsymbol{0}\}$ is removed or, inversely, we cover $\mathbb{R}^4$ with an infinity of mappings of $\mathbb{R}^2 \times \mathbb{R}^2$ (figure 2).

The transformation F is canonical and one of its generating functions reads

$$W(p_x, p_y; t, \mathscr{L}) = \mathscr{L}(\tan^{-1} p_y/p_x + N\pi) - (-1)^N t(p_x^2 + p_y^2)/2m. \tag{2.4}$$

It means

$$q_x = -\partial W/\partial p_x \qquad q_y = -\partial W/\partial p_y \qquad \chi = \partial W/\partial \mathscr{L} \qquad h = -\partial W/\partial t$$

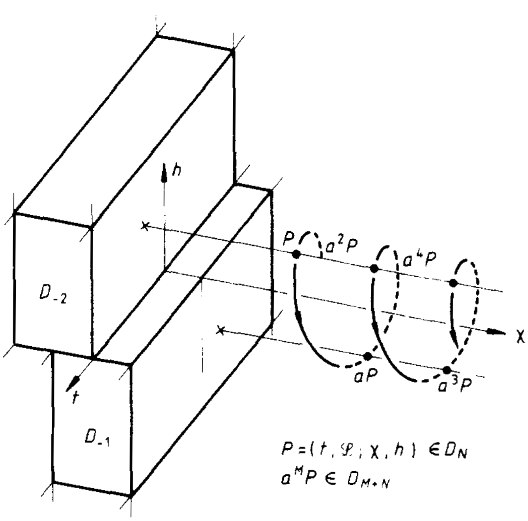

Figure 2. With the generator a, D_N is transformed into D_{N+1} by a translation of π along the χ axis and a rotation of π about the same axis.

and it is simple to calculate the Jacobian

$$\det\begin{pmatrix} \partial^2 W/\partial p_x \partial \mathscr{L} & \partial^2 W/\partial p_x \partial t \\ \partial^2 W/\partial p_y \partial \mathscr{L} & \partial^2 W/\partial p_y \partial t \end{pmatrix} = \frac{(-1)^N}{m} \neq 0$$

which says that W unfolds globally the transformation F.

In the new coordinates the canonical S transformation is very simple. It transforms the straight line $(t, \mathscr{L}; \chi, h)$, $-\infty < t < +\infty$, into the new straight line $(t', \mathscr{L}'; \chi', h')$, $-\infty < t' < +\infty$, canonically, where $h' = h$ and $\mathscr{L}' = \mathscr{L}$ with our hypothesis on the potential.

The transformation is like a gauge transformation in the energy–angular-momentum variables, and the corresponding generating function reads

$$G(t', \mathscr{L}'; \chi, h) = \chi\mathscr{L}' - ht' + 2\hbar\delta_{\mathscr{L}'}(|h|) \tag{2.5}$$

with

$$t = -\partial G/\partial h \qquad \mathscr{L} = \partial G/\partial \chi \qquad \chi' = \partial G/\partial \mathscr{L}' \qquad h' = -\partial G/\partial t'$$

where $\delta_{\mathscr{L}'}(|h|)$ is a functional of the scattering potential, as explicitly given by Narnhofer and Thirring. The derivative $2\hbar\, \partial\delta/\partial h$ is the time delay and $2\hbar\, \partial\delta/\partial\mathscr{L}'$ the deflection angle. This angle may be bigger than 2π in the case of orbiting.

3. Quantisation

Before quantising the classical variables $(t, \mathscr{L}; \chi, h)$, we have to choose a Hilbert space. Because $(t, \mathscr{L}; \chi, h) \in \mathbb{R}^4$, we suggest to work with

$$\mathscr{H} = L^2(\mathbb{R}^2, \mathrm{d}\kappa\, \mathrm{d}\varepsilon\,; \mathbb{C}).$$

We define the $|\kappa, \varepsilon\rangle$ representation of $\mathscr{H}$ by

$$\langle \kappa, \varepsilon | \kappa', \varepsilon' \rangle = \delta(\kappa - \kappa')\delta(\varepsilon - \varepsilon') \tag{3.1a}$$

$$\iint \mathrm{d}\kappa\, \mathrm{d}\varepsilon\, |\kappa, \varepsilon\rangle\langle\kappa, \varepsilon| = \mathbb{1} \tag{3.1b}$$

so that $|\kappa, \varepsilon\rangle$ form an orthonormal basis of $\mathscr{H}$ in Dirac's sense. The $|\tau, \lambda\rangle$ representation of $\mathscr{H}$ is given by

$$\langle \kappa, \varepsilon | \tau, \lambda \rangle = \frac{1}{2\pi\hbar} \exp[(\mathrm{i}/\hbar)(\kappa\lambda - \varepsilon\tau)] \tag{3.2}$$

in the (3.1) representation. We again have orthogonality and completeness:

$$\langle \tau, \lambda | \tau', \lambda' \rangle = \delta(\tau - \tau')\delta(\lambda - \lambda') \tag{3.3a}$$

$$\iint \mathrm{d}\tau\, \mathrm{d}\lambda\, |\tau, \lambda\rangle\langle\tau, \lambda| = \mathbb{1}. \tag{3.3b}$$

It is interesting to compare the 'plane' wave (3.2) in the angle and time variables with (2.5). The phase of the unitary kernel (3.2) has the Van Vleck form corresponding to the generating function (2.5) if the scattering potential is zero, i.e. for $\delta_{\mathscr{L}'}(|h|) = 0$. Hence for this transformation, the WKB form is exact.

The quantisation of the classical quantities $(t, \mathscr{L}; \chi, h)$ is now possible. We associate the operators $(T, L; K, H)$ to them. They act on $\mathscr{H}$ and in the $|\kappa, \varepsilon\rangle$ representation they are defined by

$$\langle\kappa,\varepsilon|T|\psi\rangle=\mathrm{i}\hbar\frac{\partial}{\partial\varepsilon}\langle\kappa,\varepsilon|\psi\rangle \qquad \langle\kappa,\varepsilon|L|\psi\rangle=-\mathrm{i}\hbar\frac{\partial}{\partial\kappa}\langle\kappa,\varepsilon|\psi\rangle$$
$$\langle\kappa,\varepsilon|K|\psi\rangle=\kappa\langle\kappa,\varepsilon|\psi\rangle \qquad \langle\kappa,\varepsilon|H|\psi\rangle=\varepsilon\langle\kappa,\varepsilon|\psi\rangle. \tag{3.4}$$

Because $(\kappa,\varepsilon)\in\mathbb{R}^2$, $(T,L;K,H)$ are self-adjoint and have a continuous real spectrum. Moreover, $|\kappa,\varepsilon\rangle$ are eigenfunctions of (K,H) and writing (3.4) in the $|\tau,\lambda\rangle$ representation we see that the $|\tau,\lambda\rangle$ are eigenfunctions of (T,L). We can also control that

$$[T,H]=[K,L]=\mathrm{i}\hbar$$

and that all the other commutators vanish.

Now, we should like to describe the number of turns of the projectile around the scatterer. To this end, we define three new operators

$$N=2[K/2\pi]\theta(H)+(2[(K+\pi)/2\pi]-1)\theta(-H) \tag{3.5a}$$
$$P=\sqrt{2m|H|} \tag{3.5b}$$
$$\Phi=K-N\pi \tag{3.5c}$$

where $[x]$ denotes the integer part of x.

The operator N will be very useful for the description of a possible orbiting around the potential. Clearly, (P,Φ,N) form a complete set of commuting observables and their eigenfunctions $|p,\varphi,n\rangle$ are given by

$$\langle\kappa,\varepsilon|p,\varphi,n\rangle=m^{-1/2}\delta(\varepsilon-(-1)^n p^2/2m)\delta(\kappa-\varphi-n\pi) \tag{3.6a}$$
$$\langle\tau,\lambda|p,\varphi,n\rangle=\frac{1}{2\pi\hbar}\frac{1}{\sqrt{m}}\exp\{-(\mathrm{i}/\hbar)[\lambda(\varphi+n\pi)-(-1)^n\tau p^2/2m]\}. \tag{3.6b}$$

The range of the spectrum of (P,Φ,N) is $\mathbb{R}^+\times[0,2\pi[\times\mathbb{Z}$, so the states $|p,\varphi,n\rangle$ represent a generalisation of the momentum representation. Again the phase of (3.6*b*) is a generating function, namely identical with (2.4). The $|p,\varphi,n\rangle$ representation constitutes a new orthonormal complete set, i.e.

$$\langle p,\varphi,n|p',\varphi',n'\rangle=p^{-1}\delta(p-p')\delta(\varphi-\varphi')\delta_{nn'} \tag{3.7a}$$
$$\sum_n\int_0^\infty p\,\mathrm{d}p\int_0^{2\pi}\mathrm{d}\varphi\,|p,\varphi,n\rangle\langle p,\varphi,n|=\mathbb{1}. \tag{3.7b}$$

To prove (3.7*a*) we use the partition of the identity (3.1*b*)

$$\begin{aligned}\langle p,\varphi,n|p',\varphi',n'\rangle&=\frac{1}{m}\iint \mathrm{d}\kappa\,\mathrm{d}\varepsilon\,\delta\left(\varepsilon-(-1)^n\frac{p^2}{2m}\right)\\&\quad\times\delta\left(\varepsilon-(-1)^{n'}\frac{p'^2}{2m}\right)\delta(\kappa-\varphi-n\pi)\delta(\kappa-\varphi'-n'\pi)\\&=\frac{1}{m}\delta\left(\frac{p^2}{2m}-\frac{p'^2}{2m}\right)\delta(\varphi-\varphi')\delta_{nn'}\\&=\frac{1}{p}\delta(p-p')\delta(\varphi-\varphi')\delta_{nn'}\end{aligned}$$

since $(\varphi,\varphi')\in[0,2\pi[$, $(p,p')\in\mathbb{R}^+$.

The completeness relation may be verified by taking matrix elements of (3.7*b*):

$$
\begin{aligned}
\langle\kappa,\varepsilon| & \sum_n \int_0^\infty p\,\mathrm{d}p \int_0^{2\pi} \mathrm{d}\varphi\,|p,\varphi,n\rangle\langle p,\varphi,n|\kappa',\varepsilon'\rangle \\
&= \frac{1}{m}\sum_n \int_0^\infty p\,\mathrm{d}p \int_0^{2\pi} \mathrm{d}\varphi\,\delta\left(\varepsilon-(-1)^n\frac{p^2}{2m}\right)\delta\left(\varepsilon'-(-1)^n\frac{p^2}{2m}\right) \\
&\quad \times\delta(\kappa-\varphi-n\pi)\delta(\kappa'-\varphi-n\pi) \\
&= \iint \mathrm{d}x\,\mathrm{d}y\,\delta(\varepsilon-x)\delta(\varepsilon'-x)\delta(\kappa-y)\delta(\kappa'-y) \\
&= \delta(\kappa-\kappa')\delta(\varepsilon-\varepsilon').
\end{aligned}
$$

The above calculations exhibit an isomorphism between $\mathscr{H}$ and

$$\mathscr{K} = l^2(\mathbb{Z}; L^2(\mathbb{R}^+\times[0,2\pi[, p\,\mathrm{d}p\,\mathrm{d}\varphi; \mathbb{C}))$$

which is suitable for the description of the orbiting.

In fact, $|p,\varphi,n\rangle$ is a 'basis' of $\mathscr{K}$ and the unitary operator $\mathscr{U}$ defined by $\mathscr{U}|\kappa,\varepsilon\rangle = |p,\varphi,n\rangle$ transforms a 'basis' of $\mathscr{H}$ in a 'basis' of $\mathscr{K}$.

To describe the classical mapping a, we introduce the shift operator A defined by

$$\langle p,\varphi,n|A|\psi\rangle = \langle p,\varphi,n+1|\psi\rangle. \tag{3.8}$$

This operator is obviously unitary because $n\in\mathbb{Z}$. Using the partitions of the identity (3.1*b*) and (3.3*b*) we find also

$$\langle\kappa,\varepsilon|A|\psi\rangle = \langle\kappa+\pi,-\varepsilon|\psi\rangle \qquad \langle\tau,\lambda|A|\psi\rangle = \mathrm{e}^{\mathrm{i}\lambda\pi/\hbar}\langle-\tau,\lambda|\psi\rangle. \tag{3.9}$$

We have now introduced all the mathematical formalism needed later about the Hilbert space. The reader could skip directly to § 4.

However, conceptually, we have also to discuss the configuration space. Here we encounter the difficulty that operators $\boldsymbol{Q}$ and N fail to commute. Following ideas suggested by the work of Zak (1968) we can take $\boldsymbol{Q}$ and A as a system of commuting operators. We define the $|\boldsymbol{q},\nu\rangle$ representation by

$$\langle p,\varphi,n|\boldsymbol{q},\nu\rangle = \frac{1}{2\pi\hbar}\exp[-(\mathrm{i}/\hbar)p(q_x\cos\varphi+q_y\sin\varphi)]\exp(\mathrm{i}2\pi\nu n)$$

$$(\boldsymbol{q},\nu)\in\mathbb{R}^2\times[0,1[\tag{3.10}$$

having the properties

$$\langle\boldsymbol{q},\nu|\boldsymbol{q}',\nu'\rangle = \delta(\boldsymbol{q}-\boldsymbol{q}')\delta(\nu-\nu') \tag{3.11a}$$

$$\iint \mathrm{d}^2q \int_0^1 \mathrm{d}\nu\,|\boldsymbol{q},\nu\rangle\langle\boldsymbol{q},\nu| = \mathbb{1}. \tag{3.11b}$$

We shall prove only (3.11a); with (3.7b) we obtain

$$\begin{aligned}\langle \boldsymbol{q}, \nu | \boldsymbol{q}', \nu' \rangle &= \frac{1}{(2\pi\hbar)^2} \int_0^\infty p \, \mathrm{d}p \int_0^{2\pi} \mathrm{d}\varphi \exp\{(\mathrm{i}/\hbar) p [\cos\varphi (q_x - q'_x) \\ &\quad + \sin\varphi (q_y - q'_y)]\} \sum_n \exp[\mathrm{i}2\pi n(\nu' - \nu)] \\ &= \delta(\boldsymbol{q} - \boldsymbol{q}') \sum_n \delta(\nu - \nu' + n) \\ &= \delta(\boldsymbol{q} - \boldsymbol{q}') \delta(\nu - \nu')\end{aligned}$$

where we have used the Poisson formula (A2) and $(\nu, \nu') \in [0, 1[$.

So, in the $|\boldsymbol{q}, \nu\rangle$ representation A is diagonal: using (3.8) and (3.10) we obtain

$$\langle \boldsymbol{q}, \nu | A | \psi \rangle = \mathrm{e}^{\mathrm{i}2\pi\nu} \langle \boldsymbol{q}, \nu | \psi \rangle \tag{3.12}$$

which shows that $\mathrm{e}^{\mathrm{i}2\pi\nu}$ is an eigenvalue of A.

Using (3.12) and (3.9) we obtain

$$\langle \boldsymbol{q}, \nu | A^2 | \tau, \lambda \rangle = \exp(\mathrm{i}2\pi 2\nu) \langle \boldsymbol{q}, \nu | \tau, \lambda \rangle = \exp(\mathrm{i}2\pi\lambda/\hbar) \langle \boldsymbol{q}, \nu | \tau, \lambda \rangle.$$

This means that, if the matrix elements are different from zero, then $2\nu - \lambda/\hbar$ has to be an integer. In other words, 2ν is related to the fractional part of the angular momentum in $\hbar$ units.

This achieves the complete analogy between classical and quantum descriptions. The eigenvalues $(\tau, \lambda; \kappa, \varepsilon)$ correspond to the classical variables $(t, \mathscr{L}; \chi, h)$ in the same sense as the eigenvalues of $(\boldsymbol{Q}, \boldsymbol{P})$ correspond to classical variables $(\boldsymbol{q}, \boldsymbol{p})$.

4. *S*-matrix elements and scattering amplitude

The so-called S matrix is a unitary operator which relates the asymptotic free states before and after the collision. The matrix elements may always be written

$$\langle \kappa, \varepsilon | S | \tau, \lambda \rangle = \frac{1}{2\pi\hbar} \exp[(\mathrm{i}/\hbar)(\kappa\lambda - \varepsilon\tau + 2\hbar\Delta(\tau, \lambda; \kappa, \varepsilon))]. \tag{4.1}$$

This form is chosen in analogy with (3.2) and (2.5) but it is general as long as Δ is a complex function. Without scattering potential, Δ is zero and $S = \mathbb{1}$.

For a central real potential, energy and angular momentum are conserved quantities. This means that

$$\langle \kappa, \varepsilon | S | \kappa', \varepsilon' \rangle \propto \delta(\varepsilon - \varepsilon') \qquad \langle \tau, \lambda | S | \tau', \lambda' \rangle \propto \delta(\lambda - \lambda'). \tag{4.2}$$

Calculation of these matrix elements from (4.1) tells us that Δ has to be independent of κ and τ to guarantee (4.2). The unitarity of the S matrix

$$\iint \mathrm{d}\kappa \, \mathrm{d}\varepsilon \, \langle \tau, \lambda | S^+ | \kappa, \varepsilon \rangle \langle \kappa, \varepsilon | S | \tau', \lambda' \rangle = \delta(\tau - \tau') \delta(\lambda - \lambda')$$

$$\iint \mathrm{d}\tau \, \mathrm{d}\lambda \, \langle \kappa, \varepsilon | S | \tau, \lambda \rangle \langle \tau, \lambda | S^+ | \kappa', \varepsilon' \rangle = \delta(\kappa - \kappa') \delta(\varepsilon - \varepsilon')$$

demands reality of Δ.

Since the original scattering Hamiltonian commutes with the shift operator A, we must demand that S and A commute. Using this fact and (3.9) we can write

$$\begin{aligned}\langle\kappa, \varepsilon|S|\tau, \lambda\rangle &= \langle\kappa, \varepsilon|ASA^{+}|\tau, \lambda\rangle \\ &= \exp(-\mathrm{i}\lambda\pi/\hbar)\,\langle\kappa+\pi, -\varepsilon|S|-\tau, \lambda\rangle \\ &= \langle\kappa, -\varepsilon|S|-\tau, \lambda\rangle.\end{aligned}$$

The matrix elements are invariant for the change $(\varepsilon, \tau)\mapsto(-\varepsilon, -\tau)$. This is possible iff Δ is a function of $|\varepsilon|$. Finally

$$\langle\kappa, \varepsilon|S|\tau, \lambda\rangle = \frac{1}{2\pi\hbar}\exp\,[(\mathrm{i}/\hbar)(\kappa\lambda - \varepsilon\tau + 2\hbar\delta_\lambda(|\varepsilon|))]. \tag{4.3}$$

The analogy with the classical result is complete. The exponent is identical with the classical generating function (2.5). The exact parametrisation (4.3) is defined for all real values of the angular momentum and the deflection angle is the derivative of $2\hbar\delta$ with respect to λ. Of course δ has to be calculated in a quantum way with the Schrödinger equation, but the interpretation of the result may be completely classical in the form (4.3).

Now, if we want to make a real scattering experiment we have the difficulty that neither angular momentum λ nor time τ are accessible. In fact, we prepare and detect some momentum $\boldsymbol{p}$. But in our space $\mathscr{H}$, $\boldsymbol{P}$ do not constitute a complete set of commuting observables. We have to give the sheet n. This seems to be complicated but is of great advantage: $\langle\boldsymbol{p}', n'|S|\boldsymbol{p}, n\rangle$ is the probability amplitude to measure the momentum $\boldsymbol{p}'$ after the collisions if the state was prepared in the state $|\boldsymbol{p}, n\rangle$ after an orbiting of $\frac{1}{2}(n'-n)$ turns around the potential.

A conventional detector is not sensitive to this number of turns, and by subtracting the initial beam, the probability amplitude of detection is

$$\begin{aligned}&\sum_{n'}\langle\boldsymbol{p}', n'|S-\mathbb{1}|\boldsymbol{p}, n\rangle \\ &\quad=\sum_{n'}\iiiint \mathrm{d}\kappa\;\mathrm{d}\varepsilon\;\mathrm{d}\tau\;\mathrm{d}\lambda\,\langle\boldsymbol{p}', n'|\kappa, \varepsilon\rangle\langle\kappa, \varepsilon|S-\mathbb{1}|\tau, \lambda\rangle\langle\tau, \lambda|\boldsymbol{p}, n\rangle \\ &\quad=\frac{1}{2\pi\hbar m}\sum_{n'}\delta\left((-1)^n\frac{p^2}{2m}-(-1)^{n'}\frac{p'^2}{2m}\right) \\ &\qquad\times\int \mathrm{d}\lambda\,\exp\,[-(\mathrm{i}/\hbar)\lambda(\varphi-\varphi'+(n-n')\pi)]\{\exp\,[\mathrm{i}2\delta_\lambda(p^2/2m)]-1\}.\end{aligned}$$

Contributions to the sum arise for even differences $n'-n$. Again with the Poisson formula (A1) we obtain

$$\sum_{n'}\langle\boldsymbol{p}', n'|S-\mathbb{1}|\boldsymbol{p}, n\rangle = \frac{1}{2\pi m}\,\delta\left(\frac{p^2}{2m}-\frac{p'^2}{2m}\right)\sum_{l}\exp\,[\mathrm{i}l(\varphi'-\varphi)]\{\exp\,[\mathrm{i}2\delta_{\hbar l}(p^2/2m)]-1\}. \tag{4.4}$$

In this way we obtain the scattering amplitude in the usual form as given by Henneberger (1980) for the scattering in the plane. The kinematical factor in front is also traditional. As it should be, the result is independent of the initial arbitrary chosen sheet n.

In group theoretical language, the sum on n' is the projection onto the trivial representation of the ambiguity group. This projection picks up the integer values of the angular momentum $\lambda = \hbar l$, $l \in \mathbb{Z}$. Therefore, we lost important information but this weakness is not a disease of quantum mechanics. It is the result of a lack of imagination in the detection device.

In our opinion the usual limitation to integer values of l from the beginning is a root of the difficulties of the inverse problem. We need some interpolation devices which follows from more supplementary conditions imposed on the potential, usually locality.

5. Conclusion

The main result of this work is the need to introduce a sheeted phase space for the description of scattering. The sheets are related to the number of turns around the scatterer. In this way, the angular momentum takes all real values and the complete information on the collision is contained in the phaseshift, a derivable function of energy and angular momentum.

It seems that the use of an additional Aharonov–Bohm effect as a kind of interference plate would give access to other representations of the ambiguity group. In this way, the phase would be accessible (in principle) also for non-integer values of the angular momentum.

Appendix. Poisson formula for physicists

Starting from the usual formula (Berry 1980, Spiegel 1968)

$$\sum_n \int \mathrm{d}\lambda\, F(\lambda) \exp(\mathrm{i}2\pi n\lambda) = \sum_l F(l) \tag{A1}$$

and choosing $F(\lambda) = \exp(\mathrm{i}2\pi x\lambda)$ we obtain

$$\begin{aligned}\sum_n \int \mathrm{d}\lambda\, F(\lambda) \exp(\mathrm{i}2\pi n\lambda) &= \sum_n \int \mathrm{d}\lambda\, \exp[\mathrm{i}2\pi(n+x)\lambda] \\ &= 2\pi \sum_n \delta(2\pi(n+x))\end{aligned}$$

i.e.

$$\sum_n \delta(n+x) = \sum_n \exp(\mathrm{i}2\pi nx). \tag{A2}$$

This is the translation in physicist notation of the result of Schwartz (1979) in distribution theory.

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